Florida Association of Mu Alpha Theta's

Fall 2010 Interschool

General Instructions

This is a chapter-wide team test. Anyone involved with your chapter may collaborate on this test. Only one answer sheet should be submitted per chapter. Ties will be broken by the sudden death method. You may use calculators, computers, or any other resource materials.

All submitted answers should be complete and exact. Approximations will not be accepted unless otherwise specified. Fractional answers or parts thereof should be left as improper fractions with relatively prime numerator and denominator. The acceleration due to gravity is taken to be $-9.8 m/s^2$.

Every question is worth one point unless otherwise noted. If the question is broken into parts (labeled a, b, c, etc.), then each part will be worth a corresponding fraction of the question's value.

Finally, there is a new section of the Fall Interschool encoded into the first seven questions of the test. The value of this section is determined by which school gets the farthest and will be scaled accordingly. This portion should be taken seriously, as it will be weighted heavily.

The answer sheet is to be postmarked by November 10 and is to be mailed to:

FAMAT Interschool Test
828 SW 58th Terrace
Gainesville, FL 32607

The last two questions involve the poster topic and the logo for the 2010 State Convention. The poster topic and t-shirt/program logos must be postmarked by November 10 and mailed to :

Attn. : Lisa Herron, Math Dept.
Cypress Bay High School
18600 Vista Park BLVD
Weston, FL 33332

Solutions will be available at the site listed above on November 14. Schools will have until November 21 to submit disputes via email to InterschoolDisputesFall2010@hotmail.com

Good luck and enjoy!

1. Even though this is the first question, start with question 2. You'll see that the order is important. Let $x + 7 = -138.5 - 135.7 - 132.9 - \ldots + 133.1 + 135.9 + 138.7$. What is x? (Now go to question x)

2. The cross product exists and is nontrivial in 3 and x dimensions. What is x? (Now go to question x)

3. Proper fractions with only z or y in the denominator do not yield infinite decimal expansions. $\gcd(x, z) \neq 1$. The sum of the prime factors of y and x are equal. What is x? (Now go to question x)

4. **BBB.**playworkz**.WLH/UOAMKOWU.IAH.   BIKA FX G? (OLB NL AL TCUXAFLO G)**

5. Consider the following: $f(z) = z^2 - z + 2$, $g(z) = z^3 - 4z^2 - 1$, $h(z) = \sqrt{\dfrac{10z - 27}{13}}$.
Let $x = h(g(f(h(f(g(f(2)))))))$. What is x? (Now go to question x)

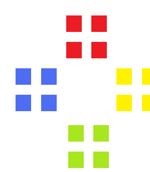
6. I want to cook six eggs. Each egg takes one minute to fry each side. However, my frying pan can only hold four eggs at a time. Let x be the fewest number of minutes in which I can complete this.

7. There are 3 red and x green balls in a basket. Two balls are drawn without replacement. The probability that the colors of the balls are the same is equal to 3/7. What is x? (Now go to question x)

---

8. Starting on January 1, 2010 and recurring every two years, a cellular phone company gives five coupons to a family of five, each redeemable for a free replacement phone. Let $x_n = 1 + .25n$ be the amount of time in years that it takes for the nth family member to break their phone upon receiving a replacement, thus using one of the coupons. Assuming that each family member has received a replacement phone on January 1, 2010, how many years after that time will a family member's phone break without the family having a coupon to replace it?

9. Sixteen people break up into four teams of four to play dodgeball: Red team, yellow team, green team, and blue team. How many distinct ways can those sixteen players be split up into four new teams of four such that each team has one member of each color?



Standing 5m away from a wall 100m tall and 10m thick, I kick a ball with an initial velocity of 60m/s.
10a. Assuming I kick the ball over the wall, let x and y be the farthest and nearest possible distance from me to the ball when it lands, respectively. What is $x + y$ (in meters)?
10b. The wall must be more than z meters thick so that for every choice of my initial kick angle the ball will not clear the wall. What is z?

11. A circle has diameter AB and chord CD. When extended, they meet at E. $\overline{EF}$ is tangent to the circle at F, $m\angle FEA = 30°$ and $m\angle CEA = 10°$. Find the measures of arcs AC, CF, FD, and DB.

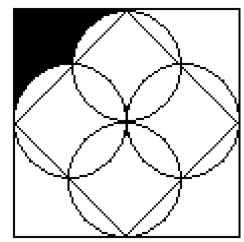
12. What is the smallest positive integer divisible by 9999 that doesn't contain the digit 9?

13. Consider the recurrence relation $a_n = 6a_{n-1} - 25a_{n-2}$ with initial conditions $a_0 = 1$, $a_1 = 6$. What are the last two digits of a_{64}?

14. Eunice is in her prime. Her age is a product of two primes p_1 and p_2. Last year, her age was also a product of two primes. The difference $p_1 - p_2$ is a product of two other primes p_3 and p_4, with the difference $p_3 - p_4$ being another prime p_5. Assuming $p_1 \neq p_2 \neq p_3 \neq p_4 \neq p_5$ and that Eunice's age is only two digits, what is $p_1 + p_2 + p_3 + p_4 + p_5$?

15. Let $S(n)$ be the sum of the digits of a positive integer n. Define $a_1 = 1$ and $a_{n+1} = S(S(a_n) + a_n)$. What is a_{2010}.

16. The large square to the right has side length 1. Determine the area of the shaded region.



17. Let B_n be the set of positive integers less than n and relatively prime to n. For which n does B_n form an arithmetic progression? (A sequence with only one term is considered an arithmetic progression)

18. Consider rectangle MNOP with width 5 and length 2. What is the average value of all line segments drawn from vertex M to sides NO and OP?

19. Suppose the point (a, b, c) lies on the sphere of radius 1 centered at the origin. Then it is true that the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ lies outside the sphere of radius r. What is the largest possible value for r?

20a. What shape is generated by revolving the line $\frac{x-3}{2} = \frac{y+1}{4} = \frac{2-z}{3}$ about $\frac{1-x}{3} = \frac{y-6}{3} = \frac{z+4}{2}$?

20b. What shape is generated by revolving the line $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z+3}{2}$ about $\frac{x-2}{2} = \frac{4-y}{5} = \frac{z-1}{3}$?

21. An equilateral triangle and regular pentagon are inscribed in a unit circle and both have a vertex in common. What is the area of the trapezoidal region formed? (Round to 3 decimal places)

22. "OIK DZARNAK NLA HQNAOAZJ; RGG AGJA HJ NLA UIZP IS CRQ."

23. Consider the six digit numbers x and y. The concatenation of x and y is divisible by the product of x and y. What is $x + y$?

24. Consider the following recurrence relation for polynomials: $p_0(x) = 1$ and if $n > 0$, $p_n(0) = 0$ and $p'_{n+1}(x) = p_n(1-x)$. Determine a formula for $p_n(1)$.

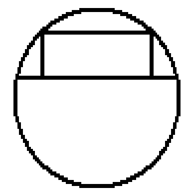
25. Consider the sequence of sets: {1}, {3, 6}, {10, 15, 21}, {28, 36, 45, 55}, … . Let S_n be the sum of the elements of the *n*th set. What is S_{100}?

26. What values of *n* make $5n^2+4$ a perfect square? Be concise.

27a. Of all cubic polynomials with integer coefficients each in the interval [-10, 10], which of them has a root that's closest to π. Assume the coefficient of x^3 is positive.
27b. Of all cubic polynomials with integer coefficients each in the interval [-10, 10], which of them has a root that's closest to *e*. Assume the coefficient of x^3 is positive.

28. Solve for *x*: $\left\lceil \frac{x-3}{2} \right\rceil = \left\lfloor \frac{x-2}{3} \right\rfloor$.

29. A unit circle has a chord of length 1.6 cutting the circle into two regions. What is the area of the largest rectangle that can be inscribed in the smaller of the two regions? (Round your answer to three decimal places)

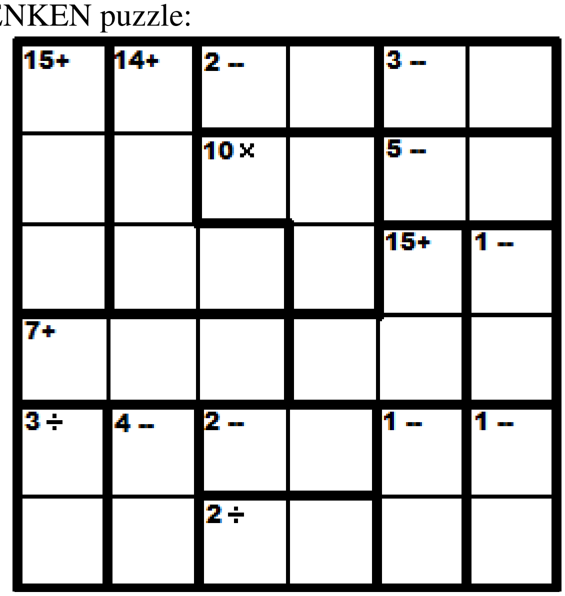


30a. What is the smallest positive integer that is not a factor of any number that has no repeated digits?
30b. What is the smallest positive prime that is not a factor of any number that has no repeated digits?

3*i*. Chris and Andy are standing on a 3-mile long bridge when a train located 2 miles away from one end of the bridge starts barreling down the track at them. Chris runs toward the end closest to the train and makes it off the bridge the instant that the train arrives at that end. Conversely, Andy runs toward the end farthest from the train and makes it off the bridge at the instant the train arrives at that end. Suppose that Andy is faster than Chris and that the speed of the train is the sum of the individual speeds of Chris and Andy. Let *x* be the distance Chris runs. What is the value of *x*? (Now go to question $\lfloor x^2 \rfloor$)

32. Determine a five digit number *n* so that $n^2+(n+1)^2+...+(n+22)^2$ is a perfect square.

33. Solve the following KENKEN puzzle:



34. At t_0, Alex is driving in his car (which is undergoing a constant acceleration) at the speed limit of 30 meters per second. Two seconds after t_0, he covers a distance of 80 meters. What is the angle that a pendulum dangling from the roof of Alex's car makes with the perpendicular to the roof?

35. Standing at the edge of the roof of his house, Dylan tosses a rock vertically upward into the air. At the moment before the rock strikes the ground 8 meters below where Dylan stands, its speed is four times that of its initial speed, *v*. What is *v*?

36.

```
  N E P T U N E
    S A T U R N
    U R A N U S
+     P L U T O
---------------
  P L A N E T S
```

37. In Sour Poker, each player receives ten cards, from which he or she makes two five card hands. When the players reveal their hands, only the worse of the two hands are compared. For a given set of ten cards, the best possible worse hand is called the sour hand. What set of ten cards produce the worst possible sour hand?

38. A chest of bananas is locked and requires multiple keys to open its multiple locks. Ten monkeys are in charge of the chest, each carrying some number of the necessary keys. Together, no group of four monkeys can open the chest with their collective keys, but any group of five can. What is the fewest locks possible for this scenario?

39. Ali bungee jumps from a tall bridge and lets out a shriek as she jumps. 1.4 seconds after falling, she hears her echo. How many meters tall is the bridge from which she jumps? Let 340m/s be the speed of sound.

40. Simplify: $\dfrac{\sin(x)+\sin(2x)+...+\sin(nx)}{\cos(x)+\cos(2x)+...+\cos(nx)}$.

41a. Is it possible to choose the signs so that $\pm1\pm2\pm3\pm4\pm...\pm99\pm100 = 2010$?
41b. Is it possible to choose the signs so that $\pm1\pm2\pm3\pm4\pm...\pm101\pm102 = 2010$?

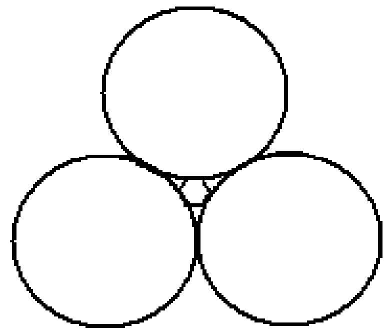
42. Find all natural numbers *n* having no zeroes for digits such that $n \cdot \bar{n} = 1000 + P(n)$, where $\bar{n}$ is the reversal of the digits of *n* and $P(n)$ is the product of the digits of *n*.

43. If the area of a triangle is an integer, then the triangle is called a *chili* triangle. What is the fourth smallest integral value of *n* for which *n*, *n* + 1, and *n* + 2 form the side lengths of a *chili* triangle.

44. **A** is an *n* by *n* matrix with the entry a_{ij} is given by $\dfrac{1}{1-4(i-j)^2}$. Consider the matrix multiplication **Ax** = **b**, where **b** is the vector of all 1's. What is the sum of the entries of **x** (in terms of n)?

45. The speed of light is given by *c*. Amanda watches as a car drives by at half the speed of light. Andy, a passenger in the car, then throws a football at half the speed of light in the same direction the car is traveling. In terms of *c*, how fast is the baseball traveling in Amanda's reference frame?

46. Hexagon ABCDEF has side length one. Three circles of equal radius are externally tangent to $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, and are externally tangent to each other. What is the radius of these circles?

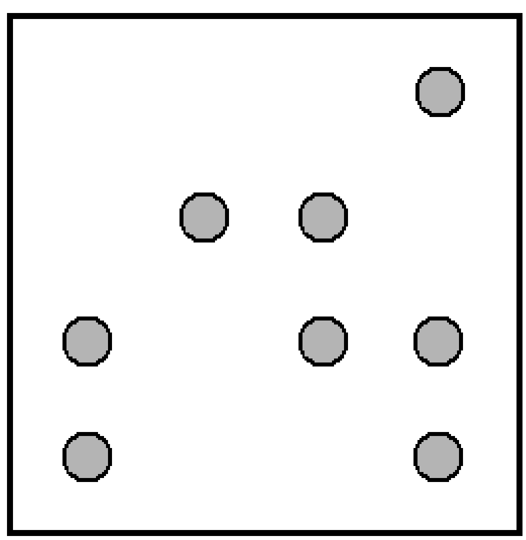


47. Consider the equation $\binom{x+1}{y} = \binom{x}{y+1}$. Determine all of the solutions (x, y) for which $100 < x + y < 1000$.

48. Find p and q such that $px^{15} + qx^{14} + 1$ is divisible by $x^2 - x - 1$.

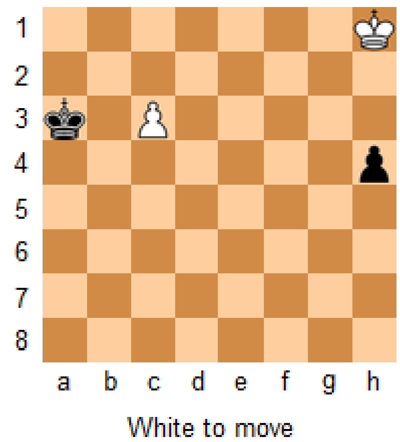
49. What does the following sequence converge to (if at all): $\frac{1}{2}, \frac{1/2}{3/4}, \frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}, \ldots$?

50. Connect the dots with a circuit consisting of line segments in such a way that each line segment drawn has a different length. A circuit is a path that finishes where it starts. It's okay if the path crosses itself, but each dot must be visited only once.



51. In the board game *Settlers of Catan*, what is the fewest number of resources necessary to achieve victory?

52. What is the outcome (white wins, black wins, or draw) of the below endgame under optimal play:



53. Grading fall interschool answer sheets can be tedious. So create an original math-related comic strip to entertain the grader, but no more than four panels please. Creativity will be rewarded and plagiarism will be penalized, so don't use that latest *xkcd* or any other. The grader reads *xkcd*. Write the comic strip on the back of your answer sheet.

54. Suggest a topic for the Poster Competition at the 2011 FAMAT State Convention.
(This question is worth zero points; however, answering may give you an advantage in the 2011 FAMAT State Convention Poster Competition.)

55. Prepare a T-shirt/test cover design for the 2011 FAMAT State Convention. This question will not count as part of the test, but will carry with it financial rewards for the winning schools. Black and white, camera ready, pictures may be placed on standard white copy paper. A second copy of the same picture with suggested colors can also be turned in. Write your school name and FAMAT ID number on the back of the design. The school whose design is selected by the FAMAT Board for the front of the State Convention t-shirt will receive two free student registrations for the State Convention while the school whose design is selected for the back of the t-shirt will receive one free student registration. Schools may submit more than one design.