Reminder: Challenges must be emailed to 2011interschool@gmail.com by November 14th.

Section 1

1a) The diagonal of the cube has length $2\sqrt{3}$ so it has edge length of 2. The volume is **8**.

1b) This is $\left(4/2\right)^2 - 2\left(1/2\right) = 4 - 1 = $ **3**.

1c) Use row operations to simplify the matrix, or just bash out – the determinant is **0**.

1d) We have that $A^2 - AB + B^2 = 60/6 = 10$, and $(A+B)^2 = 36 = A^2 + 2AB + B^2$, so $AB = \frac{36-10}{3} = 26/3$. The integral is $\frac{\sqrt{3}}{2}\left(B^2 - A^2\right) = \frac{\sqrt{3}}{2}(6)(B-A) = 3\sqrt{3}(B-A)$. We have that $(B-A)^2 = A^2 + B^2 - 2AB = 10 - 26/3 = 4/3$, so $B - A = \pm\frac{2}{\sqrt{3}}$. Then, our desired value is $\left|\pm 3\sqrt{3}\cdot\frac{2}{\sqrt{3}}\right| = $ **6**.

1e) There are 100 elements, so $A = \frac{\sum (x_i - X)^2}{100}$ and $B = \frac{\sum (x_i - X)^2}{99}$, and thus $A/B = 99/100$. The desired answer is $\left\lceil 99/100 \right\rceil = $ **1**.

Section 2

2a) Note (either via a computational device, noting a pattern, or otherwise) that $(M_n)^4$ is the *n* by *n* identity matrix. Thus, the sum of the elements (2011 1s) is just **2011**.

2b) There are 190 elements in *S*: any combination of *j* = 0,1,…,19 and *k* = 0,1,…,19 are in *S* with *j* and *k* distinct (380 possibilities), and we divide by 2 for overcounting (don't forget the 0th power!). Look for patterns or directly compute: there are 33 such elements in *S*. The probability is **33/190**.

2c) Let $x_2 = a$. We expand the sequence out to find a pattern. We see that we can express our sum as $\sum_{k=0}^{\infty}\left[(-1)^k \frac{3k+1}{2^{3k}}\right] + a\cdot\sum_{k=0}^{\infty}\left[(-1)^k \frac{3k+2}{2^{3k}}\right] + \left(\frac{a}{2} - \frac{1}{4}\right)\cdot\sum_{k=0}^{\infty}\left[(-1)^k \frac{3k+3}{2^{3k}}\right]$ $= \frac{16}{27} + \frac{40}{27}a + \frac{64}{27}\left(\frac{a}{2} - \frac{1}{4}\right) = \frac{40}{27}a + \frac{32}{27}a = \frac{72}{27}a$. The answer is $\frac{27}{a}\cdot\frac{72}{27}a = $ **72**.

2d) We can say the common difference is $x_1 - x_2 = x_2 - x_3 = \cdots = D$. Then, we can rewrite the equation as $(x_1 - x_2)(x_1 + x_2) + (x_3 - x_4)(x_3 + x_4) + \cdots = D(x_1 + x_2 + \cdots) = (x_1 + x_2 + \cdots)$, so $D = 1$. The value of $x_1 - x_{2012}$ is $2011D = $ **2011**.

2e) Write $u^2 = a + b + c - d$ with non-negative *u*, so $d = a + b + c - u^2$. Then, we can write the equation as $a^2 + b^2 + c^2 + 1 = a + b + c - u^2 + u$, or $(a - .5)^2 + (b - .5)^2 + (c - .5)^2 + (u - .5)^2 = 0$. Thus, the only real solution is $a = b = c = u = .5$, so $d = .5 + .5 + .5 - .25 = 1.25$. Hence, $100D = $ **125**.

Section 3
3a) The songs are Kryptonite by 3 Doors Down and Just the Girl by The Click Five. The desired sum is $3+5=$ **8**.
3b) The songs are Misery by Maroon 5 and Changes by 2Pac. The desired sum is $5+2=$ **7**.
3c) The songs are All the Small Things by blink-182 and Blue (Da Ba Dee) by Eiffel 65. The desired sum is $182+65=$ **247**. We will also accept **–117** if you interpret blink-182 as –182.
3d) The songs are Ghosts n Stuff by Deadmau5 and Animal I Have Become by Three Days Grace. The desired sum is $5+3=$ **8**.
3e) The songs are In Da Club by 50 Cent and From Yesterday by 30 Seconds to Mars. The desired sum is $50+30=$ **80**. We will also accept **30.50** if you interpret 50 Cent as 0.50.

Section 4
4a) No question asked.
4b) There is "No L", or "Noel" as in Christmas (Dec. 25th). 1225. (No points for this.)
4c) Snooki and Miley Cyrus were both born on Nov. 23rd. 1123. (No points for this.)
4d) This is the logo for bitly. We then go to the web address bit.ly/12251123 where we can download the mp3. (No points for this.)
4e) The songs are 1979 by The Smashing Pumpkins and 21 Guns by Green Day. The desired sum is $1979+21=$ **2000**.

Section 5
5a) **Parentheses, Exponents, Multiplication, Division, Addition, Subtraction.** Other answers relating to this are all acceptable.
5b) **Simon's Favorite Factoring Trick.** Other answers possibly acceptable.
5c) **Don't forget to carry the 1.** Other answers possibly acceptable.
5d) **Complete(ing) the square.** Other answers possibly acceptable.
5e) **Square(ing) the circle.** Other answers possibly acceptable.

Section 6
6a) Keyboard shift of "The Great Gatsby", by **F. Scott Fitzgerald.**
6b) Hexadecimal for "The Poisonwood Bible", by **Barbara Kingsolver**.
6c) ISBN for "The Meaning of it All", by **Richard Feynman**.
6d) This is the length and publishing date of the Complex Analysis book by **Walter Rudin**.
6e) ISBN shifted by 2 base 10 for "The Lion, the Witch, and the Wardrobe" by **C.S. Lewis**.

Section 7
Note that votes are only considered from schools that submitted answers to the interschool.
7a) The most number of votes was **502631** (followed by 41314, 3011, 1329, 668).
7b) The least number of votes was **0** (one school), followed by a bunch of 1s.
7c) The highest prime number of votes was **502631** (followed by 3011, 223, 19, 5).
7d) The median number of votes was **4** (4 schools at 4, 5 schools at 3 or 5).
7e) The mean number of votes was **around 15.5k**. Closest were 3011, 1329, 668, 223, 19.

Section 8
8a) **Anagram** (Google this for more about it, it's pretty neat.)
8b) **Hardy** and **Ramanujan**
8c) Some acceptable answers are **Terrance Tao** or **Akshay Venkatesh**.
8d) **Kolmogorov**
8e) **Ramanujan**

Section 9
9a) Some acceptable answers are **Princeton**, **Princeton Math Building**, **Fine Hall**, etc.
9b) This is **James Stewart's house**. (Integral House and other answers are acceptable.)
9c) This is the headquarters for **Wolfram Research**.
9d) This is a headquarters for **Casio**.
9e) This is Count von Count ("The Count") from Sesame Street (he teaches math, and is generally awesome). His castle in the show is based on Belvedere Castle in Central Park in **New York City**.

Section 10
10a) Letting $a=b=0,$ we have $f(0)=6f(0),$ so $f(0)=0.$ Then, letting $a=0,$ we have $f(b)=f(0)+4f(b)+f(0)=4f(b),$ so $f(b)=0$ for all reals. The only function that satisfies this relation is thus $f(x)=0,$ for a total of **1** function.
10b) Let Patrick's number be $0\le P\le 1.$ Then Dhyan's number *D* can range from 0 to *P* (if it exceeded *P*, then surely Patrick's number could not exceed the sum of Dhyan's number). Bishoy's number *B* can then range from 0 to $P-D.$ The probability is thus $\int_0^1\int_0^P (P-D)\,dD\,dP =$
$\int_0^1\left(P^2-\frac{P^2}{2}\right)dP = \int_0^1\left(\frac{P^2}{2}\right)dP = \frac{1^3}{6} = \frac{1}{6} \approx$ **0.17**.
10c) Because the door was opened by accident, it doesn't add any extra information (in contrast to the usual statement of the Monty Hall problem). The answer is **0.5**.
10d) Be careful: although the vertices are in clockwise order, this is a *concave* polygon (so the shoelace formula wont work here, without modification). Instead, we can find this area as the sum of the area of two trapezoids: $\frac{1}{2}(1+2)(1)+\frac{1}{2}(1+3)(2)=$ **5.5**.
10e) Let's write $z=a+bi,$ (*a* and *b* are reals), so $z^2=(a^2-b^2)+(2ab)i.$ The area of the first triangle, using the shoelace formula, is $\frac{\left|2a^2b-b(a^2-b^2)\right|}{2}=\frac{(a^2+b^2)|b|}{2}$ and the area of the second triangle is $\frac{|b|}{2}.$ Thus, $a^2+b^2=1,$ so $|z|=\sqrt{1}=$ **1**.

Section 11

11a)

Row 1:	**1, 2, 3, 4**
Row 2:	**2, 3, 4, 1**
Row 3:	**3, 4, 1, 2**
Row 4:	**4, 1, 2, 3**

11b)

Row 1:	**3, 2, 4, 1**
Row 2:	**1, 3, 2, 4**
Row 3:	**2, 4, 1, 3**
Row 4:	**4, 1, 3, 2**

11c)

Row 1:	**4, 3, 6, 5, 1, 2**
Row 2:	**6, 5, 3, 1, 2, 4**
Row 3:	**1, 4, 2, 6, 5, 3**
Row 4:	**5, 1, 4, 2, 3, 6**
Row 5:	**3, 2, 1, 4, 6, 5**
Row 6:	**2, 6, 5, 3, 4, 1**

11d)

Row 1:	**6, 5, 2, 3, 4, 1**
Row 2:	**4, 2, 3, 1, 6, 5**
Row 3:	**1, 3, 4, 5, 2, 6**
Row 4:	**2, 1, 5, 6, 3, 4**
Row 5:	**5, 4, 6, 2, 1, 3**
Row 6:	**3, 6, 1, 4, 5, 2**

11e)

Row 1:	**5, 1, 8, 7, 2, 3, 4, 6**
Row 2:	**6, 5, 4, 2, 3, 1, 8, 7**
Row 3:	**1, 2, 3, 4, 7, 5, 6, 8**
Row 4:	**2, 3, 7, 1, 8, 6, 5, 4**
Row 5:	**3, 4, 1, 6, 5, 8, 7, 2**
Row 6:	**7, 8, 2, 5, 6, 4, 3, 1**
Row 7:	**8, 7, 6, 3, 4, 2, 1, 5**
Row 8:	**4, 6, 5, 8, 1, 7, 2, 3**

Section 12

Note that statistics are only considered from schools that submitted answers to a given part.

12a) The average was 16.28, so 2/3 of the average was **10.85**.
Closest were: 12, 7.77, 2.99, 2.01, 20 (x2).

12b) The average was 12.236, so 2/3 of the average was **8.157**.
Closest, without exceeding, were: 7.77, 6, 2.99, 1.04, 1.01.

12c) The median was 10, so 2/3 of the median was **20/3 = 6.67**.
Closest were: 7.77, 9, 10, 2.99, 1.01.

12d) The third quartile was 31.67, so 2/3 of the third quartile was **21.11**.
Closest were: 20 (x3), 30, 10.

12e) The third quartile was 25, so 2/3 of the third quartile was **50/3 = 16.67**.
Closest, without exceeding, were: 15 (x2), 9, 7.77, 2.99, 1.01.

Section 13

13a) You can show that any guess is dominated by a smaller guess – you want to guess as close to 0 as possible (well, not in reality, but in a perfectly rationalizable world), so the best guess is **0.01** since we required that the number be in (0,100) and be a positive integer when multiplied by 100.

13b) Use a backwards induction logic to see that 1 will always elect to choose L. Then there is a ½ chance that the payoff will be (4,0) and a ½ chance that the payoff will be (1,1). Player 2's expected payoff is thus **0.5**.

13c) d strictly dominates c, so we eliminate c. Then, b strictly dominates a. The equilibrium is **(b,d)** or **(d,b)**.

13d) Eliminate strictly dominated mixed strategies. You will be left with the equilibrium **(z,c)** or **(c,z)**.

13e) Note that the term $-\frac{1}{100}\sum_{j\neq k} x_j$ is independent of x_k, so student *k* will just try to maximize

$x_k - \frac{1}{5}x_k^2$ (regardless of what everyone else does). This maximum occurs at $\frac{1}{2\cdot 1/5} =$ **2.5**.

Section 14

14a) See here: http://en.wikipedia.org/wiki/Passer_rating We find that $a=\left(\frac{234}{350}-.3\right)\times 5=1.84286$, $b=\left(\frac{3286}{350}-3\right)\times .25=1.59714$, $c=\left(\frac{32}{350}\right)\times 20=1.82857$, and $d=2.375-\left(\frac{6}{350}\times 25\right)=1.94643$.

Since none of these exceed 2.375, the passer rating is $\left(\frac{a+b+c+d}{6}\right)\times 100\approx$ **120**.

14b) The answer is **Miguel Cabrera** with a pop of about 0.457. The next highest is Jose Bautista with a pop of about 0.451.

14c) The answer is the **Edmonton Oilers** with a value of $3-30=-27$.

14d) There were 27 penalty kicks taken in shootouts, 19 of which were made. Thus, the desired answer is $19/27\approx$ **70%**.

14e) Make a spreadsheet with all of these values, and be sure to weight them with the number of serves (don't just average the first serve percentages). The number of first serves made was 3842 and the number of first serves attempted was 6261. The desired answer is $\frac{3842}{6261}\approx$ **61%**.

Section 15

Answers will vary dramatically. If you would like to discuss your specific code, email 2011interschool@gmail.com.

Section 16

16a) This is the number of edges, or **7**.

16b) The number of edges in a complete graph with *n* vertices is $\frac{n(n-1)}{2}$. The desired answer is $\frac{2011\cdot 2010}{2}=$ **2021055**.

16c) The row space is the column space of the transpose, which is $c_1\begin{bmatrix}2\\3\end{bmatrix}$, so it is a subset of 2-space. Thus, the value of *a* is **2**.

16d) Since $f_1(x)=\frac{9}{2}f_2(x)$, the Wronskian must be **0**. We could also compute this directly.

16e) The empty set is both open and closed. **C**.

Section 17

17a) This is $\frac{5!}{2!}=$ **60**.

17b) This is $\left(\sqrt{2}+\sqrt{3}\right)^6\approx$ **970**.

17c) The next smallest palindrome is 979, so *k* is **9**.

17d) Half of 9 is 4.5 (not an integer), so this is impossible. The probability is **0**.

17e) The ceiling function of 0 is 0. Thus, the smallest possible value of *N* is **2011**.

Section 18

18a) **1337**.

18b) There is a $1/1337$ chance that the numbers are equal – otherwise, the probability that Rob's number is greater is the same as the probability that Brandi's number is greater. Thus, $p = \frac{1-1/1337}{2} = 1336/(2\cdot 1337)$, so $Kp = 1336/2 =$ **668**.

18c) We note that $\lfloor 668/50 \rfloor = 13$. Then, we have 13 balls and 4 dividers (search "balls and urns" if you are not familiar with this method). The number of ways to distribute the balls is $\frac{17!}{13!4!} =$ **2380**.

18d) We note that $\lceil 2380/50 \rceil = 48$, so the radius of the semicircle is 24. Let's consider this to be the semicircle $y = \sqrt{24^2 - x^2}$. It should be clear that we want the endpoints of the longer base of the trapezoid to be the endpoints of the diameter of the semicircle, so the longer base has length 48. Then, the other vertices are $\left(\pm a, \sqrt{24^2 - a^2}\right)$, and the area of the trapezoid is $\frac{1}{2}(48+2a)\left(\sqrt{24^2-a^2}\right) = (24+a)\left(\sqrt{24^2-a^2}\right)$. Taking the derivative to maximize this shows that it has a maximum at $a = 12$, which gives the area as $432\sqrt{3} \approx$ **748**.

18e) We hope that there is something better than just 748 alone (1). Let these integers be $k+1, k+2, ..., k+j$. Then, we can express this sum as $(1+2+\cdots+(k+j)) - (1+2+\cdots+k)$. Thus, we need to find 748 as the difference of two triangular numbers. Searching for a list of triangular numbers, we can continue to test them for a bit, until we find the desired property: $748 = 1378 - 630 = (1+2+\cdots+52) - (1+2+\cdots+35)$. Hence, we can write $748 = 36+37+\cdots+52$, which is $52-36+1 =$ **17** consecutive integers.

Section 19

19a)

i) Rodrigo's statement means that his root is of the form $p+qi$ where $q \neq 0$ because if his root were real, Aaron and Pratik might have non-real roots and Rodrigo would not be able to make this statement. However, the person with the real root (since there can't be three non-real roots) doesn't know if Rodrigo's root is real, because that person doesn't know if all three people have real roots.

ii) Because Aaron knows that it must be with his root, this means that his root is of the form $p-qi$, since if z is a root of $f(x)$, then $\bar{z}$ must be as well. In fact, now Rodrigo and Aaron know each other's roots. Pratik now knows the form of the other two's roots, but not their specific values.

iii) Let r be Pratik's root. Then, $\frac{1}{p+qi} + \frac{1}{p-qi} + \frac{1}{r} = r + \frac{2p}{p^2+q^2}$, which simplifies to $\frac{1}{r} = r$, so $|r| = 1$. While Pratik obviously knows the value of r, Rodrigo and Aaron only know that $|r| = 1$.

iv) This gives $2p = 4|r| = 4$, so the sum of the roots is $(p+qi)+(p-qi)+r = 2p+r = 4+r$.

v) Since $f''(x) = 6x+2a$, the x-coordinate of the inflection point is $-a/3$. Because $-a$ is the sum of the roots, this means that the inflection point occurs at $x = \frac{4+r}{3}$. Because Pratik, who knows the value of r, says that this is an integer, and $|r|=1$, r must be -1. The inflection point thus occurs at $x=\frac{4-1}{3}=\mathbf{1}$.

19b) There was some ambiguity with this question. One solution: Let $A(x)=\sum_{n\geq 0} a_n x^n$ be the generating function of the sequence. Then, $A^2(x)=\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k a_{n-k}\right)x^n=\sum_{n\geq 0} x^n=\frac{1}{1-x}$. We derive

$$\binom{-1/2}{n}=\frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!}=$$

$$\frac{1\cdot 3\cdots(2n-1)}{(-2)^n n!}=\frac{(2\cdot 4\cdots 2n)(1\cdot 3\cdots(2n-1))}{(-4)^n n!n!}=\frac{\binom{2n}{n}}{(-4)^n}.$$ Then, with the binomial theorem, we have our desired sequence as $a_n=\frac{\binom{2n}{n}}{(-4)^n}$, and a computational device (i.e. Wolfram Alpha) tells us that

$\left\lfloor 10^6\cdot\left|\frac{\binom{4022}{2011}}{(-4)^{2011}}\right|\right\rfloor=\mathbf{12580}$. An alternate interpretation leads to an answer of **497** (also accepted).

19c) This is highly relevant to **Sylvester's Four-Point Problem**. By taking the limit as *n* goes to infinity via L'Hopital's Rule or a computational device, you obtain the quantity $35/12\pi^2$ which is extremely important to this problem (and should make searching for it much easier). See the internet for further exploration.

Section 20

20a) The value of *A* could range from 1 through 20 (no courses higher than 24). However, it cannot be 9 (no course 13), 13 (no course 13), 15 (no course 19), 19 (no course 19 nor 23). Clearly, *B* is determined completely by *A*. Thus, the total number of possibilities is $20-4=$ **16**.
20b) **Playpen balls** (or **ball pit** or equivalent). See various web sources.
20c) **RRT**. See here: http://www.youtube.com/watch?v=azEXrtWIR14
20d) You can use the MIT People Directory to find out that RRT lives in **Maseeh Hall**.
20e) **Alpha Delta Phi** (or **ADP** or **ADPhi**). See here: http://tech.mit.edu/V131/N39/rush.html

Section 21

Answers will vary dramatically. In general, see Wikipedia or other sources for explanations.

Section 22

22a)

+	+	-	-	+	-	+	P	+	+	sphere
-	-	-	+	+	+	-	+	-	+	point
+	-	+	-	-	-	+	+	-	+	plane
-	+	-	+	-	+	-	-	-	+	math
O	-	+	+	-	-	+	L	+	+	times
-	+	-	Y	+	-	-	-	+	-	add
-	-	N	-	+	O	+	M	+	-	sum
+	+	-	+	-	+	I	-	+	+	divide
-	A	+	-	+	-	+	+	-	+	graph
-	+	-	+	-	+	-	+	L	+	equal

s	s	p	l	p	m	e	o	c	p
q	i	r	i	r	i	v	d	o	i
u	d	i	n	i	n	e	d	n	
a	e	m	e	s	u	n		e	
r		e		m	s				
e		s							

The answer is **polynomial**. (+ denotes horizontal word, - denotes vertical word).

Section 23

23a) Examining the words shows that each adjacent pair of letters is superimposed. There are 325 ways to superimpose two different letters, which suggests base conversion. In fact, this is base 325 arithmetic, where an AB combination equals 0, AC equals 1, BC equals 25, and so on.
Using this revelation to evaluate the final example, we get four pairs of letters: QS, IU, DI, and KN. Putting these in a reasonable order to get a phrase out spells the answer **squid ink**.