2011 FAMAT State Convention
Interschool Test

Each of the following 50 questions is worth one point. Each of the following sections, listed in alphabetical order, contains 5 questions. The questions are in no particular order. The last page of this test contains an answer sheet, on which you must write your simplified, exact answers to turn in at the convention. You may use any resources, provided that you work only among members and sponsors of your school's Mu Alpha Theta chapter. Good luck, and have fun!

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1 Advanced Mathematics

1. There is a conjecture in number theory that states that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds for all n, where p_n is the nth prime number. Give the name of this conjecture, stated by a Romanian mathematician, that has been shown to be true for the first 10^{16} primes.

2. If x_n satisfies $\left(1+\frac{1}{n}\right)^{n+x_n} = e$, find $\lim_{n\to\infty} x_n$.

3. Let $F_0(x) = \ln x$, and for $n \geq 0$, let $F_{n+1} = \int_0^x F_n(t)\,dt$. Evaluate $\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}$.

4. Evaluate $\int_{\mathbb{R}} \frac{\cos x}{1+x^2}\,dx$.

5. Name the mathematician most commonly associated with the mathematical/integration method used to evaluate the integral in #4.

2 Algebra

6. The letters 'ALGEBRA' can be arranged in $\frac{7!}{2!} = 2520$ different ways. But only one of these ways makes an anagram of ALGEBRA that, when split into two words, is something very common in science class. Which is that (give the letters as a 7-letter "word")?

7. You could try to answer question 6 by listing out the 2520 different arrangements in alphabetical order, but that would be very inefficient. If you did do this, though, the first arrangement would be AABEGLR, the second would be AABEGRL, and so on. What would be the 2011th arrangement?

8. Find the value of

$$\prod_{k=1}^{\prod_{k=1}^{\prod_{k=1}^{\cdot^{\cdot^{\cdot}}} \frac{\sqrt[5]{600}}{k}} \frac{\sqrt[5]{600}}{k}} \frac{\sqrt[5]{600}}{k}$$

9. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, $f(1) = 1$, $f(2) = 4$, and $f(3) = 9$. Find $f(6) + f(-2)$.

10. A geometric sequence of real numbers, with first term and common ratio both not equal to zero, has the properties that the product of the first four terms equals the sum of the first three terms, and the sum of the first four terms equals the average of the first three terms. What is the second term of this sequence?

3 Calculus

11. How many distinct *circular* arrangements are there of the letters 'CALCULUS'?

12. Find the value of $k < 0$ such that the tangent lines to the parabola $y = x^2$ that pass through $(0, k)$ are perpendicular.

13. Find $\frac{d}{dx}[f(x)]$ at $x = 2$ given that

$$f'(x) = \begin{cases} 2 - x & x \leq 0 \\ 2 & 0 < x < 2 \\ x & x \geq 2 \end{cases}$$

14. Let R be the region of points such that $x^2 + y < z,\ \ 0 < z < k$, and $y > 0$. Find, in terms of k, the volume of R.

15. In triangle ABC, $BC = 4$ cm, $AC = 3$ cm, and $\angle C$ is increasing at a rate of 5 degrees per second. Find the rate of change of the area of ABC, in cm^2/second, when $\angle C = 60$ degrees.

4 Codes

16. LZW 2011 XSESL KLSLW UGFNWFLAGF AK TWAFY ZWDV AF GJDSFVG, XDGJAVS. AF OZSL UALQ OSK AL ZWDV LWF QWSJK SYG?

17. Consider the sequence 2011, 121021, 111211101211, 31123110111221,... Give the next term of this sequence.

18. Here are a few members of a famous group. Give the name of this group.

$$\frac{\sqrt{2mE}}{k} \quad \alpha \quad \frac{d}{t} \quad \frac{nRT}{P} \quad v' \quad \frac{PV}{nT} \quad rt$$

$$(\mathbb{R} \cup \overline{\mathbb{R}}) \quad \omega \quad \lambda \quad \upsilon \quad \frac{F}{a} \quad (\delta - 2) \quad \sqrt{-1} \quad \frac{F}{m}$$

$$rt \quad \frac{c}{e} \quad \sqrt{x^2 + y^2} \quad \frac{d}{r} \quad \frac{F}{g} \quad \omega \quad \upsilon \quad \theta$$

19. Ub qgR ARrw qukk rgua twE;A BruibK NY kog RGWR xibcwbruib vw gwks.

20. ONXE UXENIUXEHDHXG HQ ZXUVWQ ZVJ OJHEHGP ENI KWVEI "ZD PUBUPQ C AIJQ USZD ZKD DZGQW AIJQN, C LDIWZG ODKQW DW US HQSQWCE CSM ODKQW KGCZQBQW USZD ZKD ODKQWN DL ZGQ NCRQ PQSDRUSCZUDS CJDBQ ZGQ NQADSP UN URODNNUJEQ, CSP U GCBQ CNNIWQPEM LDISP CS CPRUWCJEQ OWDDL DL ZGUN, JIZ ZGQ RCWHUS UN ZDD SCWWDK ZD ADSZCUS UZ."?

5 Games

21. You are playing a game with two piles of coins: one with 20 coins, and the other with 30 coins. On a given turn, you take some positive number of coins from one pile, and then your opponent takes some positive number of coins from one of the piles. If you go first, and the winner is whoever takes the last coin(s), who will win this game assuming both you and your opponent play with the best possible strategy? State "you" for you or "opponent" for your opponent.

22. Consider the game in question 21. What is your best possible first move? Give your answer in the form (pile, coins). For instance, if you want to take 5 coins from the 20 coin pile, give the answer $(20, 5)$.

23. Let's play a game: consider $f(x) = -x^2 + x + 2$ over the domain $[-1, 1]$. I'm going to let you choose a value of $x \in [-1, 1]$, but then I will choose which interval I want; that is, if you choose $x = a$, I can take the interval $[-1, a)$ leaving you with $[a, 1]$, or I could take the interval $(a, 1]$ leaving you with $[-1, a]$. A random value of $f(x)$ will be chosen, and the winner is whoever's interval contains the value of x that produced this value of $f(x)$; note, however, that each value of x is not equally likely to occur since it is the value of $f(x)$ that is being chosen at random. To maximize your probability of winning this game, what value of x should you choose, rounded to the nearest hundredth, assuming that I will play with the best possible strategy?

24. Let's play a game: I'm going to pick a number, x, randomly chosen such that $x \in [0, 2011]$. Then you must correctly guess this value of x to win the game. What is the probability that you will win this game?

25. Consider a square table of dimensions $n \times n$. June and Emily are playing a game as follows: at the beginning of the game, all of the cells are empty, and the players take turns placing a coin in one of the cells. In each move, the player can put a coin in any cell that is not adjacent to a cell that already has a coin in it (note: cells are defined as adjacent if they share an edge). The winner is whoever makes the last move (at which point all open cells are adjacent to a cell with a coin). Assume that June and Emily both play with the best possible strategies. If June goes first, and they take turns placing coins in the $n \times n$ table, for how many values of $n \in \{1, 2, 3, 4, ..., 2011\}$ will she win this game?

6 Geometry

26. A lattice point in a rectangular coordinate system is defined as a point both of whose coordinates are integers. Find the number of lattice points in the region defined by $|x| + |y| < 2011$.

27. Consider quadrilateral $ABCD$ (with vertices in that order) such that the side lengths AB, BC, CD, and DA will be chosen at random and without replacement from $\{2, 3, 5, 8, 13, 21\}$. What is the probability that A and C will be obtuse angles?

28. Consider regular octagon $ABCDEFGH$. Let T be the triangle bounded by the segments AE, BF, and CH. What is the ratio of the area of T to the area of $ABCDEFGH$?

29. It is well known that, if $A(n)$ denotes the area of a regular $n-$gon with apothem of length 1, then $\lim_{n\to\infty} A(n) = \pi$. Find the smallest integer n such that $|A(n) - \pi| \leq \frac{1}{2011}$.

30. A group of mathematicians is lost in a forest which has the shape of an infinite strip with a uniform width of 1 mile. They do not know where they are in relation to either side of the forest. Conveniently, however, they can take a path of k miles that will ensure that they exit the forest no matter where they currently are. Find the minimum possible value for k.

7 Guessing

31. Guess a number. If it matches my number, you will receive one point.

32. Consider $f(x) = \sqrt{-x^2 + x + 2}$, and let $a \in [0, 1]$ be a number I have chosen at random. You win a game if you choose a value of $x \in [0, 1]$ such that $|f(a) - f(x)| < \frac{1}{2011}$. What is the best guess you could make?

33. Choose a world capital. If it matches one on my list of 5, you will receive one point.

34. Name an album by Green Day, Nirvana, or the Red Hot Chili Peppers. If it matches the album I have chosen by one of these bands, you will receive one point.

35. In a certain book I recently read, the words on the first page use all but 3 letters. Choose a letter, and if it matches one of these 3, you will receive one point.

8 Pre-Calculus

36. Non-degenerate isosceles triangle ABC has vertex angle measuring x radians and sides of lengths $\sin x$, $\sqrt{\sin x}$, and $\sqrt{\sin x}$. What is the area of ABC?

37. Let $f(n) = 2^0 + 2^1 + 2^2 + ... + 2^n$. If $\log_2(f(n) + 1) = 9$, find the number of zeroes at the end of the decimal (base 10) expansion of $(10n)!$.

38. The roots of $f(x) = x^4 + 2x^3 - 3x^2 + ax + b$, where $\{a, b\} \in \mathbb{R}$, are $\sin\alpha$, $\cos\alpha$, $\tan\beta$, and 1. Find $|\sec\beta|$.

39. Given that matrix $\mathbf{M}$ is a 2 by 2 matrix of real numbers in which exactly two elements are 0, and $\mathbf{M}^4 = \mathbf{M}$, find all possible values of $\det\mathbf{M}$.

40. A square matrix is defined such that the element in the xth row and yth column is $\dfrac{x}{(y+1)^x}$. If $S(n)$ is the sum of the elements in such a matrix with dimensions $n \times n$, find $\lim_{n\to\infty} S(n)$.

9 Trivia

41. During the episodes that aired during April 2010, what state was represented by the most contestants on Jeopardy? Note: repeat contestants only count once, and celebrities don't count.

42. Give the name of this character:

43. What school hosted the 2006 FAMAT January Invitational?

44. Who wrote the Geometry tests for the 2010 FAMAT March Regional?

45. What actor is most widely associated with 3566678889776556356667888977656 (these numbers are a code for something)?

10 Statistics

46. Find the exact variance of the sample data set {2,4,6,8,10,12,...,1000}.

47. Let X and Y be independent random variables such that $\mu_X = 15$, $\sigma_X^2 = 3$, $\mu_Y = 10$, and $\sigma_Y^2 = 4$. Consider the distribution of the random variable $(X + Y)$. What is the $z-$statistic for the observation $(X + Y) = 24$?

48. A baseball player makes 5 plate appearances, in which he strikes out once, hits two singles, hits one home run, and is walked once. What is his slugging percentage for these 5 plate appearances?

49. A fair coin is tossed 2011 times. The probability that there are at least 2005 consecutive heads is p. What is $p \cdot 2^{2011}$?

50. Ten fair, six-sided dice are rolled. The probability that the sum of the 10 numbers showing on the dice is 35 is p. What is $p \cdot 6^{10}$?

ANSWERS

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